

GENERIC BERNSTEIN-SATO POLYNOMIAL ON AN IRREDUCIBLE AFFINE SCHEME

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ABSTRACT. Given p polynomials with coefficients in a commutative unitary integral ring \mathcal{C} containing \mathbb{Q} , we define the notion of a generic Bernstein-Sato polynomial on an irreducible affine scheme $V \subset \text{Spec}(\mathcal{C})$. We prove the existence of such a non zero rational polynomial which covers and generalizes previous existing results by H. Biosca. When \mathcal{C} is the ring of an algebraic or analytic space, we deduce a stratification of the space of the parameters such that on each stratum, there is a non zero rational polynomial which is a Bernstein-Sato polynomial for any point of the stratum. This generalizes a result of A. Leykin obtained in the case $p = 1$.

INTRODUCTION AND MAIN RESULTS

Fix $n \geq 1$ and $p \geq 1$ two integers and $v \in \mathbb{N}^p$. Let $x = (x_1, \dots, x_n)$ and $s = (s_1, \dots, s_p)$ be two systems of variables. Let \mathbf{k} be a field of characteristic ¹ 0. Let $\mathbf{A}_n(\mathbf{k})$ be the ring of differential operators with coefficients in $\mathbf{k}[x] = \mathbf{k}[x_1, \dots, x_n]$ and \mathcal{D} (resp. \mathcal{O}) be the sheaf of rings of differential operators (resp. analytic functions) on \mathbb{C}^n for which we denote by \mathcal{D}_{x_0} (resp. \mathcal{O}_{x_0}) the fiber in x_0 .

Let $f = (f_1, \dots, f_p)$ be in $\mathbf{k}[x]^p$ (resp. $\mathcal{O}_{x_0}^p$) and consider the following functional identity:

$$b(s)f^s \in \mathbf{A}_n(\mathbf{k})[s] \cdot f^{s+v},$$

(resp. $\mathcal{D}_{x_0}[s]$ instead of $\mathbf{A}_n(\mathbf{k})[s]$) where $f^{s+v} = f_1^{s_1+v_1} \dots f_p^{s_p+v_p}$. This identity takes place in the free module generated by f^s over $\mathbf{k}[x, \frac{1}{f_1 \dots f_p}, s]$ (resp. $\mathcal{O}_{x_0}[\frac{1}{f_1 \dots f_p}, s]$).

The set of such $b(s)$ is an ideal of $\mathbf{k}[s]$ (resp. $\mathbb{C}[s]$). This ideal is called the (global) Bernstein-Sato ideal of f (resp. local Bernstein-Sato ideal in x_0) and we denote it by $\mathcal{B}^v(f)$ (resp. $\mathcal{B}_{x_0}^v(f)$). When $p = 1$, this ideal is principal and its monic generator is called the Bernstein polynomial associated with f . Historically, I.N. Bernstein [Be] introduced the (global) Bernstein polynomial and proved its existence (i.e. the fact that it is not zero). J.E. Björk [Bj] has given the proof in the analytic case. Let us cite also M. Kashiwara [K] who proved, moreover, the rationality of the roots of the local Bernstein polynomial. For $p \geq 2$, the algebraic case can be easily treated in the same way as for $p = 1$. For the analytic case, the proof of the non nullity of $\mathcal{B}_{x_0}^v(f)$ is due to C. Sabbah ([S1] and [S2]). Let us also cite A. Gyoja [Gy] who proved that $\mathcal{B}_{x_0}^v(f)$ contains a non zero *rational* polynomial. The absolute Bernstein-Sato polynomial naturally leads to the notion of a generic Bernstein-Sato polynomial which we shall explain in what follows.

Let \mathcal{C} be a unitary commutative integral ring with the following condition:
For any prime ideal $\mathcal{P} \subset \mathcal{C}$ and for any $n \in \mathbb{N} \setminus \{0\}$, we have:

$$n \in \mathcal{P} \Rightarrow 1 \in \mathcal{P}.$$

¹all the fields considered in this paper are of characteristic 0

This condition is equivalent to the fact that for any $\mathcal{P} \subset \mathcal{C}$, the fraction field of \mathcal{C}/\mathcal{P} is of characteristic 0. Note that this condition is satisfied if and only if there exists an injective ring morphism $\mathbb{Q} \hookrightarrow \mathcal{C}$.

We shall see \mathcal{C} as the ring of coefficients or parameters. Indeed, let $f = (f_1, \dots, f_p)$ in $\mathcal{C}[x]^p = \mathcal{C}[x_1, \dots, x_n]^p$.

Let us denote by $\mathbf{A}_n(\mathcal{C})$ the ring of differential operators with coefficients in $\mathcal{C}[x]$, that is the \mathcal{C} -algebra generated by x_i and ∂_{x_i} ($i = 1, \dots, n$) where the only non trivial commutation relations are $[\partial_{x_i}, x_i] = 1$ for $i = 1, \dots, n$ (hence \mathcal{C} is in the center of $\mathbf{A}_n(\mathcal{C})$).

We denote by $\text{Spec}(\mathcal{C})$ (resp. $\text{Specm}(\mathcal{C})$) the set of prime (resp. maximal) ideals of \mathcal{C} which is the spectrum of \mathcal{C} (resp. the maximal spectrum). For an ideal $\mathcal{I} \subset \mathcal{C}$, we denote by $V(\mathcal{I}) = \{\mathcal{P} \in \text{Spec}(\mathcal{C}); \mathcal{P} \supset \mathcal{I}\}$ the affine scheme defined by \mathcal{I} and $V_m(\mathcal{I}) = V(\mathcal{I}) \cap \text{Specm}(\mathcal{C})$. Remark that we shall only work with the closed subsets of $\text{Spec}(\mathcal{C})$ and forget the sheaf structure of a scheme.

We are going to introduce the notion of a generic Bernstein-Sato polynomial of f on an irreducible affine scheme $V = V(\mathcal{Q}) \subset \text{Spec}(\mathcal{C})$ (that is when \mathcal{Q} is prime).

So let \mathcal{Q} be a prime ideal of \mathcal{C} and suppose that none of the f_j 's is in $\mathcal{Q}[x]$.

The main result of this article is the following.

Theorem 1. *There exists $h \in \mathcal{C} \setminus \mathcal{Q}$ and $b(s) \in \mathbb{Q}[s_1, \dots, s_p] \setminus 0$ such that*

$$hb(s)f^s \in \mathbf{A}_n(\mathcal{C})[s]f^{s+v} + \left(\mathcal{Q}[x, \frac{1}{f_1 \dots f_p}, s]\right)f^s.$$

Such a $b(s)$ is called a (rational) generic Bernstein-Sato polynomial of f on $V = V(\mathcal{Q})$ (see the notation and the remark below).

In the case where $p = 1$, the generic and relative (not introduced here) Bernstein polynomial has been studied by F. Geandier in [Ge] and by J. Briançon, F. Geandier and P. Maisonobe in [Br-Ge-M] in an analytic context (where f is an analytic function of x). In [Bi] (see also [Bi2]), H. Biosca studied these notions with $p \geq 1$ in the analytic and the algebraic context (that which we are concerned with) and proved that when

- $\mathcal{C} = \mathbb{C}[a_1, \dots, a_m]$ or
- $\mathcal{C} = \mathbb{C}\{a_1, \dots, a_m\}$ and

$\mathcal{Q} = (0)$ so that V is smooth and equal to \mathbb{C}^m or $(\mathbb{C}^m, 0)$, we have a generic Bernstein-Sato polynomial. It does not seem straightforward to adapt her proof to the case where $\mathcal{Q} \neq (0)$ (i.e. when V is singular). Let us also say that she did not mention the fact that the polynomial she constructed is rational even though a detailed study of her proof shows that it is. As it appears, our main result covers and generalizes the previous existing results in this affine situation.

Notation. Let \mathcal{P} be a prime ideal of \mathcal{C} . For c in \mathcal{C} , denote by $[c]_{\mathcal{P}}$ the class of c in the quotient \mathcal{C}/\mathcal{P} and $(c)_{\mathcal{P}} = \frac{[c]_{\mathcal{P}}}{1}$ this class viewed in the fraction field of \mathcal{C}/\mathcal{P} . We naturally extend these notations to $\mathcal{C}[x]$, $\mathbf{A}_n(\mathcal{C})$ and $\mathcal{C}[x, \frac{1}{f_1 \dots f_p}, s]$.

Remark. Using these notations, we can see that the polynomial $b(s)$ of theorem 1 is a Bernstein-Sato polynomial of $(f)_{\mathcal{P}}$ for any $\mathcal{P} \in V(\mathcal{Q}) \setminus V(h)$. This justifies the name of a generic Bernstein-Sato polynomial on $V(\mathcal{Q})$.

As an application of theorem 1, we obtain some consequences :

Corollary 2. *Fix a positive integer d and a field \mathbf{k} .*

For each $j = 1, \dots, p$, take $f_j = \sum_{|\alpha| \leq d} a_{\alpha,j} x^\alpha$ with $\alpha \in \mathbb{N}^n$ and $a_{\alpha,j}$ an indeterminate.

Take $a = (a_{\alpha,j})$ for $|\alpha| \leq d$ and $j = 1, \dots, p$ such that we see $f = (f_1, \dots, f_p)$ in $\mathbf{k}[a][x]^p$. Denote by m the number of the $a_{\alpha,j}$'s.

Then there exists a finite partition of $\mathbf{k}^m = \cup W$ where each W is a locally closed subset of \mathbf{k}^m (i.e. W is a difference of two Zariski closed sets) such that for any W , there exists a polynomial $b_W(s) \in \mathbb{Q}[s_1, \dots, s_p] \setminus 0$ such that for each a_0 in W , $b_W(s)$ is in $\mathcal{B}^v(f(a_0, x))$.

Remark.

- This corollary generalizes to the case $p \geq 2$ the main result of A. Leykin [L] and J. Briançon and Ph. Maisonobe [Br-Mai] in the case $p = 1$.
- There is another way to generalize these results: Given a well ordering $<$ on \mathbb{N}^p compatible with sums, it is possible to prove the existence of a partition $\mathbf{k}^m = \cup W$ into locally closed subsets with the following property: For any W , there exists a finite subset $G_W \subset \mathbf{k}[a][x]$ such that for any $a_0 \in W$, the set $G_W(a_0)$ is a $<$ -Gröbner basis of the Bernstein-Sato ideal $\mathcal{B}^v(f(a_0, x))$, see [Br-Mai] and [Ba].

Proof of Corollary 2. We remark that we can give the same statement as in corollary 2 for any algebraic subset $Y \subset \mathbf{k}^m$ as a space of parameters. The statement of corollary 2 will then follow from the proof of this more general statement, that we shall give by an induction on the dimension of Y . If $\dim Y = 0$, the result is trivial. Suppose $\dim Y \geq 1$. Write $Y = V_m(Q_1) \cup \dots \cup V_m(Q_r)$ where the Q_i 's are prime ideals in \mathbf{k}^m (we identify the maximal ideals of $\mathbf{k}[a]$ and the points of \mathbf{k}^m). For each i , let $h_i \in \mathbf{k}[a] \setminus Q_i$ and $b_i(s) \in \mathbb{Q}[s] \setminus 0$ be the h and $b(s)$ of theorem 1 applied to Q_i . Now, write

$$Y = \left(\bigcup_{i=1}^r V_m(Q_i) \setminus V_m(h_i) \right) \cup Y',$$

with $Y' = \bigcup (V_m(Q_i) \cap V_m(h_i))$ for which $\dim Y' < \dim Y$. Apply the induction hypothesis to Y' . We obtain that Y is a union (not necessarily disjoint) of locally closed subsets V such that for each V there exists $b_V(s) \in \mathbb{Q}[s] \setminus 0$ which is in $\mathcal{B}^v(f(a_0, x))$ for any $a_0 \in V$. Let us show now how to obtain the announced partition. Let B be the set of the obtained polynomials b_V 's. Set $B = \{b_1, \dots, b_e\}$. For any $i = 1, \dots, e$, let E_i be the set of the V 's for which $b_i = b_V$. Put

- $W_1 = \bigcup_{V \in E_1} V$,
- $W_2 = \left(\bigcup_{V \in E_2} V \right) \setminus \left(\bigcup_{V \in E_1} V \right)$,
- \vdots
- $W_e = \left(\bigcup_{V \in E_e} V \right) \setminus \left(\bigcup_{V \in E_1 \cup \dots \cup E_{e-1}} V \right)$.

Note that some of the W_i 's may be empty. The set $\{(b_1, W_1), \dots, (b_e, W_e)\}$ gives a partition $Y = \cup W_i$ in a way that $b_i \in \mathcal{B}^v(f(a_0, x))$ for any $a_0 \in W_i$. \square

Corollary 3. Take $f_1(a, x), \dots, f_p(a, x) \in \mathcal{O}(U)[x]$ where $\mathcal{O}(U)$ denotes the ring of holomorphic functions on a open subset U of \mathbb{C}^m .

Then there exists a finite partition of $U = \cup W$ where each W is an (analytic) locally closed subset of U (i.e. each W is a difference of two analytic subsets of U) such that for any W , there exists a rational non zero polynomial $b(s)$ which belongs to $\mathcal{B}^v(f(a_0, x))$ for any $a_0 \in W$.

Remark. As it will appear in the proof, we have the same result if we replace $\mathcal{O}(U)$ by $\mathbb{C}\{a_1, \dots, a_m\}$ or $\mathbf{k}[[a_1, \dots, a_m]]$ (\mathbf{k} being an arbitrary field).

Proof. Let us write $f_j(a, x) = \sum g_{\alpha,j}(a)x^\alpha$ where $g_{\alpha,j} \in \mathcal{O}(U)$. Let m be the number of the $g_{\alpha,j}$'s and let us introduce m new variables $b_{\alpha,j}$. Consider the (analytic) map $\phi : U \ni a \mapsto (b_{\alpha,j} = g_{\alpha,j}(a))_{\alpha,j} \in \mathbb{C}^m$ where \mathbb{C} is a fixed arbitrary field. Now apply corollary 2 to this situation. Let $\mathbf{k}^m = \cup W$ be the obtained partition and for any W , let $b_W \in \mathbb{Q}[s]$ be the polynomial given in 2. Now apply ϕ^{-1} . This gives a partition $U = \cup \phi^{-1}(W)$. Since ϕ is analytic, the sets $\phi^{-1}(W)$ are locally closed analytic subsets of U . It is then clear that for any W and $a_0 \in \phi^{-1}(W)$, we have $b_W \in \mathcal{B}^v(f(a_0, x))$. \square

PROOF OF THE MAIN THEOREM

In order to prove theorem 1, we shall first prove the following.

Theorem 4. Let \mathbf{k} be a field and $f \in \mathbf{k}[x]^p$. Then $\mathcal{B}^v(f) \cap \mathbb{Q}[s]$ is not zero.

Note that in [Br], the author proved (for $p = 1$) that the global Bernstein polynomial has rational roots for any field \mathbf{k} of characteristic zero. The proof of 4 will use the following propositions.

Proposition 5. Let \mathbf{K} be a subfield of a field \mathbf{L} . Suppose that $f \in \mathbf{K}[x]^p$. Let $b(s) \in \mathbf{K}[s]$ be such that $b(s)f^s \in \mathbf{A}_n(\mathbf{L})[s]f^{s+v}$. Then

$$b(s)f^s \in \mathbf{A}_n(\mathbf{K})[s]f^{s+v}.$$

Proof. The proof is inspired by [Br] in which the case $p = 1$ is treated. As \mathbf{L} is a \mathbf{K} -vector space, let us take $\{1\} \cup \{l_\gamma; \gamma \in \Gamma\}$ as a basis so that $\mathbf{L}[x, s, \frac{1}{f_1 \dots f_p}]f^s$ is a free $\mathbf{K}[x, s, \frac{1}{f_1 \dots f_p}]$ -module with $\{f^s\} \cup \{l_\gamma f^s; \gamma \in \Gamma\}$ as a basis. Now let P be in $\mathbf{A}_n(\mathbf{L})[s]$ such that $b(s)f^s = Pf^{s+v}$. We decompose $P = P_0 + P'$ where $P_0 \in \mathbf{A}_n(\mathbf{K})[s]$ and P' has its coefficients in $\bigoplus_{\gamma \in \Gamma} \mathbf{K} \cdot l_\gamma$. Now, we have:

$$b(s)f^s = P_0f^{s+v} + P'f^{s+v},$$

with $b(s)f^s$ and P_0f^{s+v} in $\mathbf{K}[x, s, \frac{1}{f_1 \dots f_p}]f^s$ and $P'f^{s+v}$ in $\bigoplus_{\gamma \in \Gamma} \mathbf{K}[x, s, \frac{1}{f_1 \dots f_p}]l_\gamma f^s$. By identification, we obtain:

$$b(s)f^s = P_0f^{s+v}.$$

\square

Proposition 6. ([Br] and [Br-Mai]) Given $f \in \mathbb{C}[x]^p$, we have :

- (1) The set $\{\mathcal{B}_{x_0}^v(f); x_0 \in \mathbb{C}^n\}$ is finite.
- (2) $\mathcal{B}^v(f)$ is the intersection of all the $\mathcal{B}_{x_0}^v(f)$ where $x_0 \in \mathbb{C}^n$.

Proof of theorem 4. We shall divide the proof into two steps:

- (a) First, suppose that $\mathbf{k} = \mathbb{C}$. By [S1], [S2] and [Gy], as mentioned in the introduction, each $\mathcal{B}_{x_0}^v(f)$ contains a non zero rational polynomial. By the previous proposition, we can take a finite product of these polynomials and obtain a rational polynomial in $\mathcal{B}^v(f)$.
- (b) Now suppose that \mathbf{k} is arbitrary. Let c_1, \dots, c_N be all the coefficients that appear in the writing of the f_j 's and consider the field $\mathbf{K} = \mathbb{Q}(c_1, \dots, c_N)$. There exist $e_1, \dots, e_N \in \mathbb{C}$ and an injective morphism of fields $\phi : \mathbf{K} \rightarrow \mathbb{C}$ such that $\phi(c_i) = e_i$ for any i . We denote by the same symbol ϕ the natural extension of ϕ from $\mathbf{K}[x]$ to $\mathbb{C}[x]$ and from

$\mathbf{A}_n(\mathbf{K})[s]$ to $\mathbf{A}_n(\mathbb{C})[s]$. Now, consider in $\mathbb{C}[s]$ the Bernstein-Sato ideal $\mathcal{B}^v(\phi(f))$ (where $\phi(f) = (\phi(f_1), \dots, \phi(f_p))$). Using the result of case (a), there exists $b(s) \in \mathbb{Q}[s] \setminus 0$ that belongs to $\mathcal{B}^v(\phi(f))$. So we have a functional equation:

$$b(s)\phi(f)^s = P \cdot \phi(f)^{s+v},$$

where $P \in \mathbf{A}_n(\mathbb{C})[s]$. By proposition 5, we can suppose $P \in \mathbf{A}_n(\phi(\mathbf{K}))[s]$. Apply ϕ^{-1} to this equation. Since $b(s) \in \mathbb{Q}[s]$, $\phi^{-1}(b(s)) = b(s)$, thus we obtain:

$$b(s)f^s = \phi^{-1}(P) \cdot f^{s+v}.$$

In conclusion $b(s)$ is in $\mathcal{B}^v(f)$. □

Now we dispose of a sufficient material to give the

Proof of theorem 1. By theorem 4, there exists a non zero rational polynomial $b(s)$ in $\mathcal{B}^v((f)_{\mathcal{Q}})$. Hence, we have the following equation:

$$b(s)\left(\frac{[f]_{\mathcal{Q}}}{1}\right)^s = \frac{[U(s)]_{\mathcal{Q}}}{[h]_{\mathcal{Q}}} \cdot \left(\frac{[f]_{\mathcal{Q}}}{1}\right)^{s+v},$$

where $U(s) \in \mathbf{A}_n(\mathbb{C})[s]$ and $h \in \mathcal{C} \setminus \mathcal{Q}$. It follows that:

$$h b(s)f^s - U(s) \cdot f^{s+v} \equiv 0 \pmod{\mathcal{Q}}$$

in $\mathcal{C}[x, \frac{1}{f_1, \dots, f_p}, s]f^s$. Since $f_1 \cdots f_p \notin \mathcal{Q}[x]$ and \mathcal{Q} is prime, we obtain:

$$h b(s)f^s - U(s) \cdot f^{s+v} \in \mathcal{Q}[x, \frac{1}{f_1 \cdots f_p}, s]f^s.$$

□

This article is a more general and simplified version of some results of my thesis [Ba].

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